equally distant from each other." This assumes that a line which is everywhere equidistant from a straight line must be itself straight, but in Bolyai's geometry this line is a curve, the well-known *equidistantial*, while in Lobachevski one of the first theorems is: "The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another."

In §62, in place of a proof for the fundamental theorem: "If two parallels are cut by a transversal the alternate angles are equal," Mr. Gore gives the flat *petitio principii*: "The lines AB and CD, being parallel, have the same direction. The lines EG and GH, being in one and the same straight line, are similarly directed. That is, the angles EGB and GHD have sides with the same direction; therefore the differences of their directions are equal." ! ! !

"§127. A circle is a plane figure"!

"§138. Two circumferences are tangent to each other when they are tangent to a straight line at the same point"! What about two circles with only one point in common, and the straight tangent to one at that point different from the straight tangent to the other at that point?

Page 64, "Up to the present time it has been assumed that any needful line or combination of lines could be drawn, and the question has not arisen as to the possibility of drawing these lines *with accuracy ! ! !*

In order to show that any required combination of lines, angles, or parts of lines or angles fulfilled the required conditions, principles were needed long before they could be demonstrated"! Mr. Gore has perhaps never looked into a copy of Euclid. Beyond his three postulates, hypothetical constructions are neither necessary nor admissible.

There is one good purpose that this book may subserve, that is to show how absolutely essential is a knowledge of non-Euclidean geometry.

Austin, Texas, 1900.

EXPRESSION OF RIEMANN'S P FUNCTION AS A DEFINITE INTEGRAL.

By W. E. HEAL.

Consider the differential equation of Riemann's P function, namely,

$$(x-a)(x-b)(x-c)\frac{d^2y}{dx^2}+(x-a)(x-b)(x-c)\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}\right.$$

$$\left.+\frac{1-\gamma-\gamma'}{x-c}\right)\frac{dy}{dx}+\left(\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}\right.$$

$$\left.+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right)y=0 \quad \ldots . \ (1),$$

where α, α', β, β', γ, γ' are real quantities and we have $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Let $V=(x-a)^{\alpha}(x-b)^{\beta}(x-c)^{\gamma}u^{-(\alpha'+\beta'+\gamma')}$

$$(1-u)^{-(\alpha'+\beta+\gamma')}[(a-b)(x-c)+(b-c)(x-a)u]^{-(\alpha+\beta+\gamma)}.$$

Then $y_1=\int_0^1 Vdu$ is an integral of equation (1).

For substituting this value for y in the equation (1) the left member becomes

$$\int_0^1\Big[(x-a)(x-b)(x-c)\left\{\frac{(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}\right.$$

$$-\frac{(\alpha+\beta+\gamma)\left(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\right)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}$$

$$+\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right)\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)$$

$$\left.+\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)^2-\left[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\right]\right\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\Big]Vdu.$$

Since $(x-a)(x-b)(x-c)\left\{\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right)\right.$

$$\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)+\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)^2$$

$$\left.-\left[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\right]\right\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}$$

$$=(\alpha+\beta+\gamma)[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c],$$

this result may be written,

$$(\alpha+\beta+\gamma)\int_0^1\Big[(x-a)(x-b)(x-c)\left\{\frac{(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}\right.$$

$$-\frac{\left(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\right)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}\Bigg\}$$

$$+[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c]\,Vdu$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)$$

$$\int_0^1 \frac{\{(\alpha+\beta+\gamma')(a-b)(x-c)+(\alpha'+\beta+\gamma)(b-c)(x-a)u^2 -[(a-b)(1+\alpha-\alpha')(x-c)+(b-c)(1+\gamma-\gamma')(x-a)]u\}\,Vdu}{[(a-b)(x-c)+(b-c)(x-a)u]^2}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\int_0^1 \frac{d}{du}\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}_0^1$$

which is identically zero if the integral has a meaning. That the integral may not become infinite at the limits we must have

$$1-(\alpha'+\beta'+\gamma)=\alpha+\beta+\gamma'>0,$$
$$1-(\alpha'+\beta+\gamma')=\alpha+\beta'+\gamma>0.$$

It is also clear that

$$y_2=\int_0^{-\infty} Vdu, \quad y_3=\int_1^{+\infty} Vdu,$$

also satisfy the differential equation. For $y=y_2$ we must have

$$\alpha+\beta+\gamma'>0, \quad \alpha'+\beta+\gamma>0.$$

And for $y=y_3$,

$$\alpha+\beta'+\gamma>0, \quad \alpha'+\beta+\gamma>0.$$

In equation (1) write

$$a=-1, \; b=1/\varepsilon, \; c=+1,$$
$$\alpha=\alpha'=\gamma=\gamma'=0, \; \beta=-n, \; \beta'=n+1, \; \lim\varepsilon=0,$$

and we find after some reductions the differential equation for zonal spherical harmonics, viz:

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{n(n+1)y}{1-x^2}=0.$$

(Craig's *Linear Differential Equations*, page 192.) In this case

$$y_1=\int_0^1\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_2=\int_0^{-\infty}\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_3=\int_1^{+\infty}\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}}.$$

In y_1 we must have n negative and numerically less than unity. For y_3 the values n may have are the same as for y_1. In y_2 we may have n any, negative, real number.

In equation (1) write

$$\begin{aligned}
a&=0,\ b=1,\ c=-1,\\
\alpha&=\tfrac{1}{2}\{(k-1)+\sqrt{[4n(n+k-1)+(k-1)^2]}\},\\
\alpha'&=\tfrac{1}{2}\{(k-1)-\sqrt{[4n(n+k-1)+(k-1)^2]}\},\\
\beta=\gamma&=\tfrac{1}{4}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},\\
\beta'=\gamma'&=\tfrac{1}{4}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},
\end{aligned}$$

and we have the differential equation,

$$\frac{d^2y}{dx^2}+\frac{2x^2+k-2}{x(x^2-1)}\frac{dy}{dx}+\frac{[n(n+k-1)(x^2-1)-m(m+k-2)x^2]y}{x^2(x^2-1)^2}=0\ldots\ldots\ldots(3),$$

which transformed by the substitution $x=1/t$ becomes the equation for spherical harmonics of rank k, namely,

$$\frac{d^2y}{dt^2}-\frac{kt}{1-t^2}\frac{dy}{dt}+\frac{[n(n+k-1)(1-t^2)-m(m+k-2)]y}{(1-t^2)^2}=0\ldots\ldots\ldots(4),$$

(Craig, page 195.)

The definite integral appears to be too complicated to be of much use except in special cases.

I. Let $k=1$.

$\therefore\ \alpha=n,\ \alpha'=-n,\ \beta=\gamma=\tfrac{1}{2}m,\ \beta'=\gamma'=\tfrac{1}{2}(1-m)$.

Equation (4) becomes for this case,

$$\frac{d^2y}{dt^2}-\frac{t}{1-t^2}\frac{dy}{dt}+\frac{[n^2(1-t^2)-m(m-1)]y}{(1-t^2)^2}=0 \quad \ldots(5).$$

We have,

$$y_1=\int_0^1(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du.$$

In all these integrals we have

$$n+\tfrac{1}{2}>0,$$

and in y_2, y_3 we must also have

$$m>n.$$

II, Let $k=2$.
$\therefore\ \alpha=n+1,\ \alpha'=-n,\ \beta=\gamma=\frac{1}{2}m,\ \beta'=\gamma'=-\frac{1}{2}m.$
Equation (4) becomes

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[n(n+1)(1-t^2)-m^2]y}{(1-t^2)^2}=0 \ldots (6),$$

which is (Craig. page 194) the eqûation of the *associated function*, $P_{n,m}$, of the first kind, of degree n and order m.

We have

$$y_1=\int_0^1(1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du.$$

In these integrals we must have

$$n+1>0,$$

and y_2, y_3 we also have

$$m>n.$$

If in (6) we put $n=\mu-\frac{1}{2}$ we have,

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[(\mu^2-\frac{1}{4})(1-t^2)-m^2]y}{(1-t^2)^2}=0\ldots.(7),$$

which is (Craig, page 194) the differential equation for Hicks' Toroidal functions.

We have for this case

$$y_1=\int_0^1(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

where $\mu+\frac{1}{2}>0$, and in y_2, y_3 $m>\mu-\frac{1}{2}$.

FORECASTING THE CENSUS RETURNS.

By JAMES S. STEVENS, Professor of Physics, The University of Maine, Orono, Maine.

Now that the government of the United States is about to take another census, we occasionally see in the newspapers forecasts of the population. Sometimes these forecasts are mere guesses, but there is a method by which one can make these estimates scientifically, and if they fail to come out right it is the fault of the people rather than the method.

All physical laws may be divided into two classes—rational and empirical. The free fall of a body, the swinging of a pendulum, and most of the laws of heat and electricity illustrate the first class. If a body falls one space the first second it will fall three the second and five the third. These laws may easily be embodied into formulae which contain no arbitrary constants. On the other hand, such problems as the relation between temperature and depth below the